Spontaneous Symmetry Breaking :

Spin-one Color Superconductor

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Spontaneous Symmetry Breaking (SSB)

- Hamiltonian has a symmetry, ground state does not respect that symmetry.

Landau-Ginzburg theory of phase transition $\Longrightarrow$ (G, H) , $H \subset G$

G $\downarrow$ Symmetry system

H $\downarrow$ Symmetry ground state (Symmetry group G broken)

$\Downarrow$

Different phases

$\Downarrow$

Different pair (G, H)

Order Parameter and SSB

- Concept of order parameter by Landau in the theory of phase transition.

Physical system exhibits SSB

$\swarrow$ Identify suitable order parameter (ϕ)

$\Downarrow$

- observations
- physical insight

$\searrow$ Computation of the order parameter

$\Downarrow$

Write most general potential G-invariant in terms of the order parameter

Group Action On The Order Parameter Space

- G compact lie Group, M differentiable manifold (points are values of the order parameter).

- Action of G on M : $\varphi : G \times M \rightarrow M$
$$(g, \phi) \rightarrow g\phi$$

- Isotropy subgroup of ϕ : $H_\phi = \{ g \in G \mid g\phi = \phi \}$

Physics : transformations left unbroken when the order parameter takes value ϕ

Group Action : Orbits

- Orbit of G through ϕ : $G(\phi) = \{ g\phi \mid g \in G \}$

$\downarrow$

Physics : Broken symmetry transformation

- Potential V invariant under group action :

$$V(\phi) = V(g\phi) \quad \forall \ \phi \in M, \ g \in G$$

$\downarrow$

V is a function on the orbits

$\downarrow$

Minimization of V on M $\Longrightarrow$ Minimization of V on the space of orbits

Isotropy Subgroups and Stratum

- Points in the same orbit $\Longrightarrow$ Conjugate isotropy subgroups :

$$H_{g\phi} = g H_\phi g^{-1}$$

- Set of all points with isotropy subgroup conjugate to H_ϕ . Stratum

  - Points of the same symmetry "class" : same unbroken subgroup, same symmetry breaking pattern

  - Phase diagram: associated with a phase , phase transition : ϕ moves from one stratum to another

Stationary Points Of G-invariant Potentials

- **Theorem (Michel)**: Let G be a compact Lie Group acting smoothly on the real manifold M and let $\phi \in M$. Then the orbit $G(\phi)$ is critical, that is, every smooth real G-invariant function on M is stationary on $G(\phi)$ if and only if $G(\phi)$ is isolated in its stratum, that is there is a neighborhood U_ϕ of ϕ such that $U_\phi \cap S(\phi) = G(\phi)$.

  - No assumption on the form of the potential

  - **Conjecture**: absolute minima of $V \Rightarrow$ orbits with maximal isotropy subgroups. → Not true in general

- **Inert States**: Stationary States of any G-invariant function.

Minimization Of The Potential

- Simplest case : order parameter transforms in an irreducible representation of the symmetry group. ①

- Most general G-invariant potential up to fourth order①:

$$V(\phi) = -\frac{1}{2} m^2 \|\phi\|^2 + \frac{1}{4} \sum_{\alpha} \lambda_\alpha \underbrace{P_\alpha^{(4)}(\phi)}_{\text{Fourth-order invariant in } \phi}$$

$P_1^{(4)}(\phi) = (\|\phi\|^2)^2$:

$$\boxed{V(\phi) = -\frac{1}{2} m^2 \|\phi\|^2 + \frac{1}{4} (\|\phi\|^2)^2 \left[\lambda_1 + \sum_{\alpha \neq 1} \lambda_\alpha A_\alpha(\phi) \right]}$$

$A_\alpha(\phi) = P_\alpha^{(4)}(\phi) / P_1^{(4)}(\phi) \Rightarrow$ orientation of the condensate

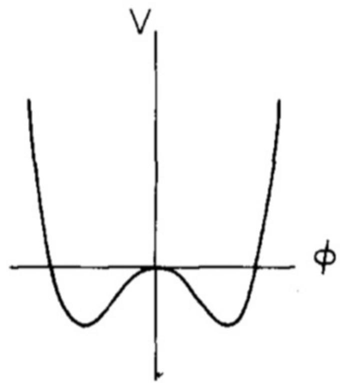


Fig1. General shape for a fourth degree potential for one irrep

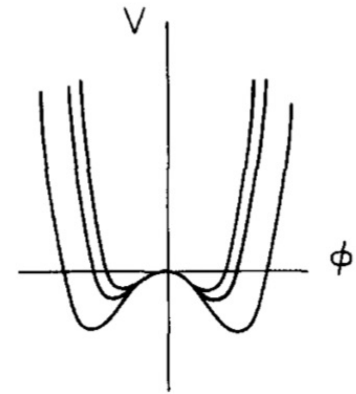


Fig2: Location of the directional minimum as the direction changes

- Minimization with respect to ϕ is equivalent to successive minimization with respect to $\|\phi\|$ and A_α.

- For fixed values of the angles $\frac{\partial V}{\partial \|\phi\|^2}$ yields :

$$\boxed{\|\phi\|^2_{min} = \frac{m^2}{\lambda_1 + \sum_{\alpha \neq 1} \lambda_\alpha \mathcal{A}_\alpha}}$$

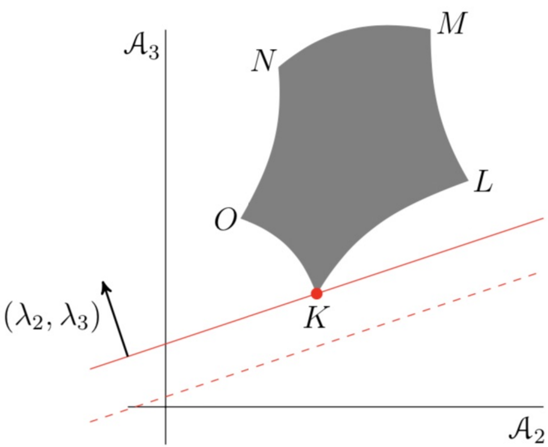
$$\boxed{V_{min}(\phi) = -\frac{1}{4} m^2 \|\phi\|^2_{min}}$$

- Absolute minima $V \Rightarrow$ Maximize $\|\phi\|_{min} \Rightarrow$ Minimize $\sum_{\alpha \neq 1} \lambda_\alpha \mathcal{A}_\alpha$

- For simplicity take three independent quartic invariants $P^{(4)}_\alpha(\phi)$

$\Downarrow$

There are 2 angles : $\mathcal{A}_2, \mathcal{A}_3$ span target space



Shape : Depends on symmetry group and representation

Not Depend on λ_α

- let's denote $E(\mathcal{A}_\phi) = \sum_{\alpha \neq 1} \lambda_\alpha \mathcal{A}_\alpha(\phi)$

$$\Downarrow$$

Set of constant $E(\mathcal{A}_\phi)$ is in the $(\mathcal{A}_2, \mathcal{A}_3)$-plane represented by straight lines

- Value of $E(\mathcal{A}_\phi)$ too low $\Rightarrow$ does not intersect target space

$$\Downarrow$$

No ϕ for this $E(\mathcal{A}_\phi)$

- $\nearrow E(\mathcal{A}_\phi) \Rightarrow$ touches target space for the first time

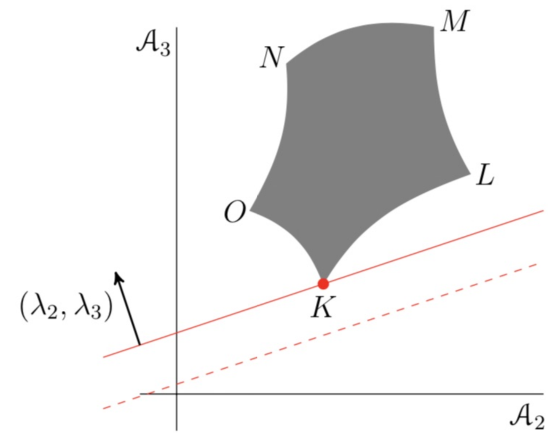
$$\Downarrow$$

Absolute minimum of $E(\mathcal{A}_\alpha)$

$$\Downarrow$$

Absolute minimum of $V(\phi)$

- Varying the couplings $\lambda_\alpha \Rightarrow$ Scan the whole phase diagram.



- Shape of order parameter does not depend on λ_1

- Any value of (λ_2, λ_3) groundstate (minimum energy state) is represented by point on the boundary of target space.

- **Boundary concave** : (λ_2, λ_3) change smoothly $\Rightarrow$ ground state changes abruptly

- **Boundary convex** : (λ_2, λ_3) change smoothly $\Rightarrow$ ground state changes continuously

Example: Spin-One Color Superconductor

- Viable candidate for the ground state of cold dense quark matter.

- Objective: phases of the groundstate of the spin-one color supercon-ductor

1_ Identify Suitable Order Parameter

- Order parameter can be represented by 3x3 complex matrix Δ_{ai} which transforms as:

$$\boxed{\Delta \rightarrow U \Delta R}$$

$$U \in SU(3) \times U(1) \equiv U(3)_L^{(2)} \quad , \quad R \in SO(3)_R$$

- Classification of all possible inequivalent forms of the order parameter based on two claims:

Theorem 1: The order parameter can always be brought to the form:

$$\Delta = \begin{pmatrix} \Delta_1 & i a_3 & -i a_2 \\ -i a_3 & \Delta_2 & i a_1 \\ -i a_1 & i a_2 & \Delta_3 \end{pmatrix} \quad \circledast$$

with real parameters Δ_j, a_j being an hermitian matrix

Theorem 2: Let the order parameter have the above form and $U \in U(3)_L$ and $R \in SO(3)_V$. Then:

$$U R^T \Delta R = \Delta$$

if and only if $U\Delta = \Delta$, $R^T \Delta R = \Delta$.

$\circledast$ Invariance under $G \equiv U(3)_L \times SO(3)_R \Rightarrow$ 18 independent variables reduce to 6 independent variables

2- Classify Forms of Δ that leave some continuous subgroups of G unbroken

- This analysis can be simplified by theorem 2, one can separately investigate invariance under $U(3)_L$ ③ and $SO(3)_V$.

Invariance under $SO(3)_V$

- Δ is hermitian $\Rightarrow$ $\Delta = S + iA$
- Action of $SO(3)_V$ on Δ : $R^T (S + iA) R = R^T S R + i R^T A R$

$\downarrow$

To compute the isotropy subgroup of $SO(3)_V$, compute for A and S separatly

- S is symmetric and real $\Rightarrow$ diagonalizable

$$S = \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix}$$

- $A = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}$

  - Then $A_{ij} = \varepsilon_{ijk} a_k$
  - Matrix A is basically determined by a vector $\vec{a} = (a_1, a_2, a_3)$

Statement : $R^T A R = R \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $R \in SO(3)$

- Isotropy subgroup of A $\Rightarrow$ rotations that leave $\vec{a}$ invariant.

- $\Delta = S + i A$ $\Rightarrow$ To leave Δ invariant it has to leave S and A invariant

$\Downarrow$

Common symmetry of S and A

Statement : S possesses the same symmetry as A if and only if this quadratic surface is axially symmetric with axis given by $\vec{a}$.

- S has this symmetry $\Rightarrow$ $S_{ij} = \alpha \delta_{ij} + \beta a_i a_j$

$\Downarrow$

Since S is diagonal :
- $\beta = 0$
- at most one of the a_i's $\neq 0$

5

Oblate 4	Cylindrical	ε	A
$\mathrm{SO}(2)_\mathrm{V}$	$\mathrm{SO}(2)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{L}$	$\mathrm{SO}(2)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{L}$	$\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SO}(2)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{L}$
$\begin{pmatrix} \Delta_1 & +ia & 0 \\ -ia & \Delta_1 & 0 \\ 0 & 0 & \Delta_2 \end{pmatrix}$	$\begin{pmatrix} \Delta & +ia & 0 \\ -ia & \Delta & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \Delta_1 & +\mathrm{i}\Delta_1 & 0 \\ -\mathrm{i}\Delta_1 & \Delta_1 & 0 \\ 0 & 0 & \Delta_2 \end{pmatrix}$	$\begin{pmatrix} 1 & +\mathrm{i} & 0 \\ -\mathrm{i} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
CSL	Polar	N_1	N_2
$\mathrm{SO}(3)_\mathrm{V}$	$\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SO}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{L}$	$\mathrm{U}(1)_\mathrm{L}$	$\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{L}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z_1 & z_2 & z_3 \end{pmatrix}$

- Oblate

$$\vec{a} = (0, 0, a) \quad \Rightarrow \quad A = \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$S_{ij} = \alpha \delta_{ij} + \beta a_i a_j \quad \Rightarrow \quad S_{11} = \alpha \qquad S_{22} = \alpha \qquad S_{33} = \alpha + \beta$$

$$S = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha+\beta \end{bmatrix}$$

$$\Delta = S + iA = \begin{bmatrix} \alpha & ia & 0 \\ -ia & \alpha & 0 \\ 0 & 0 & \alpha+\beta \end{bmatrix} = \begin{bmatrix} \Delta_1 & ia & 0 \\ -ia & \Delta_1 & 0 \\ 0 & 0 & \Delta_2 \end{bmatrix}$$

$\Rightarrow$ Isotropy subgroup are the rotation matrices that leave $(0,0,a)$ invariant

$\Downarrow$

Rotation of a plane
is represented by $\boxed{SO(2)_v}$

- In the first line : Δ more specific $\Rightarrow$ more symmetry $\Rightarrow$ bigger isotropy subgroup

- A, CSL and polar have strata with a fixed representative (not depend on a_j, Δ_j)

$\Downarrow$

Strata has one orbit $\Rightarrow$ Inert states

3_ Decide which of these phases occupy a part of the phase diagram

- One writes the most general $G = U(3)_L \times SO(3)_R$ invariant potential.

$$\boxed{V(\Delta) = -\frac{1}{2} m^2 \sqrt{P_1^{(4)}(\Delta)} + \frac{1}{4} \sum_{\alpha=1}^{3} \lambda_\alpha P_\alpha^{(4)}(\Delta)}$$

- There are three independent quartic invariants that have the following expressions:

$$P_1^{(4)}(\Delta) = [\mathrm{Tr}(\Delta\Delta^+)]^2$$

$$P_2^{(4)}(\Delta) = \mathrm{Tr}(\Delta\Delta^+\Delta\Delta^+)$$

$$P_3^{(4)}(\Delta) = \mathrm{Tr}(\Delta\Delta^T(\Delta\Delta^T)^+)$$

- Minimum of the potential $\Rightarrow$ vertices of the target space

$\Downarrow$

Shape of the target space depends on $P_\alpha^{(4)}$

- Let's consider the following four theorems which are just inequalities satisfied by the three independent quartic invariants

$\Downarrow$

inequalities determine the target space

Theorem 3: The invariants $P_1^{(4)}$ and $P_2^{(4)}$ satisfy the following inequalities:

$$\frac{1}{3} P_1^{(4)} \le P_2^{(4)} \le P_1^{(4)}$$

The first inequality is saturated iff Δ is of type CSL the second is saturated iff Δ has rank 1.

Theorem 4. The invariants $P_2^{(4)}$ and $P_3^{(4)}$ satisfy the following inequalities.

$$0 \leq P_3^{(4)} \leq P_2^{(4)}$$

The first inequality is saturated iff Δ is of type A the second is saturated iff Δ is real.

Theorem 5: The invariants $P_1^{(4)}$, $P_2^{(4)}$, $P_3^{(4)}$ satisfy the following inequality:

$$\frac{2}{3} P_1^{(4)} \leq P_2^{(4)} + P_3^{(4)}$$

The inequality is saturated iff Δ is of the type oblate with $\Delta_2 = \sqrt{\Delta_1^2 + a^2}$.

Theorem 6 : Let $P_3^{(4)} \leq \frac{1}{9} P_1^{(4)}$ Then the invariants $P_1^{(4)}$, $P_2^{(4)}$ and $P_3^{(4)}$ satisfy the following inequality.

$$\sqrt{P_1^{(4)}} \leq \sqrt{P_3^{(4)}} + \sqrt{P_2^{(4)} - P_3^{(4)}}$$

The inequality is saturated iff Δ is of type ε

- For the potential in question one has that :

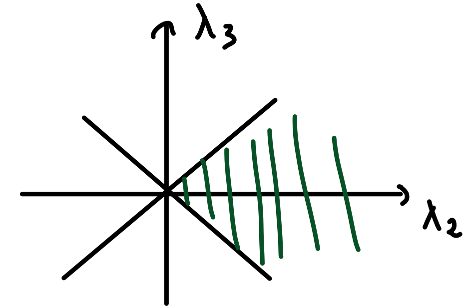
$$\sqrt{P_{1\,min}^{(4)}} = \frac{m^2}{\lambda_1 + \lambda_2 A_2 + \lambda_3 A_3} \qquad V_{min}(\phi) = -\frac{1}{4} m^2 \sqrt{P_{1\,min}^{(4)}}$$

$$A_2 = P_2^{(4)} / P_1^{(4)} \qquad A_3 = P_3^{(4)} / P_1^{(4)}$$

- Interested in minimizing $\lambda_2 A_2 + \lambda_3 A_3$, but first let's analyse

$$\boxed{\lambda_2 P_2^{(4)} + \lambda_3 P_3^{(4)}}$$

- $\lambda_2 + \lambda_3 > 0 \quad , \quad \lambda_2 > \lambda_3$



Using theorem 4 and theorem 5 :

$$\lambda_2 P_2^{(4)} + \lambda_3 P_3^{(4)} = \frac{1}{2}(\lambda_2+\lambda_3)\underbrace{\left(P_2^{(4)} + P_3^{(4)}\right)}_{\geq \frac{2}{3} P_1^{(4)} \text{ (theorem 5)}} + \frac{1}{2}\underbrace{(\lambda_2 - \lambda_3)}_{>0}\underbrace{\left(P_2^{(4)} - P_3^{(4)}\right)}_{\geq 0 \text{ (theorem 4)}}$$

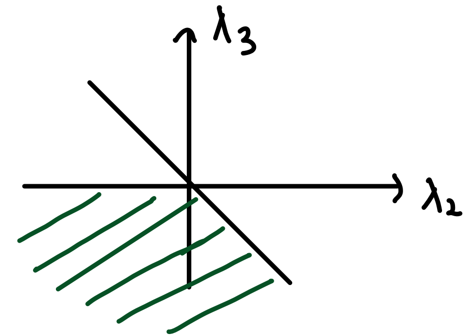
$$\geq \frac{1}{3}(\lambda_2+\lambda_3) P_1^{(4)}$$

- This is an equality iff $\cdot\, P_2^{(4)} = P_3^{(4)}$

$$P_2^{(4)} + P_3^{(4)} = \frac{2}{3} P_1^{(4)} \Rightarrow \boxed{\begin{aligned} P_2^{(4)} &= \frac{1}{3} P_1^{(4)} \\ P_3^{(4)} &= \frac{1}{3} P_1^{(4)} \end{aligned}}$$

$\Rightarrow$ By theorem 3 phase CSL

- $\lambda_2 + \lambda_3 < 0 \quad , \quad \lambda_3 < 0$



Using theorem 3 and theorem 4 :

$$\lambda_2 P_2^{(4)} + \lambda_3 P_3^{(4)} = (\lambda_2 + \lambda_3) P_2^{(4)} + \underbrace{\lambda_3}_{<0} \underbrace{\left(P_3^{(4)} - P_2^{(4)} \right)}_{\leq 0 \text{ (theorem 4)}}$$

$$\geq \underbrace{(\lambda_2 + \lambda_3)}_{<0} P_2^{(4)} \geq (\lambda_2 + \lambda_3) P_1^{(4)} \quad \text{(theorem 3)}$$

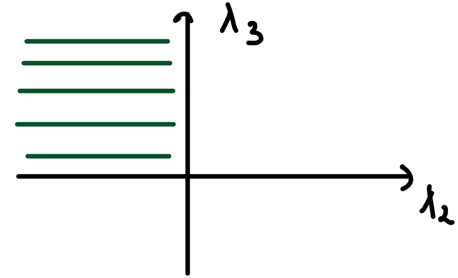
- This is an equality iff . $P_2^{(4)} = P_3^{(4)} = P_1^{(4)}$

$\Rightarrow$ By theorem 3 and theorem 4 Δ has to be real and have rank one

$\Downarrow$

Polar phase

- $\lambda_2 < 0\ ,\quad \lambda_3 > 0$



Using theorm 3 and theorm 4 :

$$\lambda_2 P_2^{(4)} + \underbrace{\lambda_3}_{>0} P_3^{(4)} \geq \lambda_2 P_2^{(4)} \geq \lambda_2 P_1^{(4)}$$

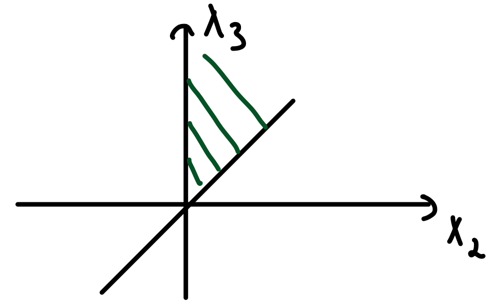
(theorm 3)

- This is an equality iff : $P_3^{(4)} = 0$

$$P_1^{(4)} = P_2^{(4)}$$

$\Rightarrow$ By theorm 4 phase A

- $\lambda_3 > \lambda_2 > 0$



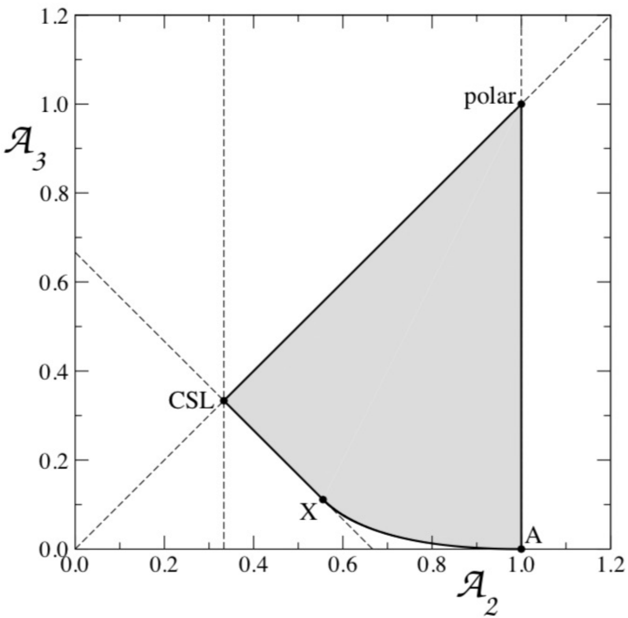
- Using theorem 6 and Cauchy inequality one arrives at the conclusion that the phase is ε 6

$\Rightarrow$ For the different values of λ_2, λ_3 using the inequalities one was able to minimize

$$\lambda_2 P_2^{(4)} + \lambda_3 P_3^{(4)} \Rightarrow \text{minimize } \lambda_2 \mathcal{A}_2 + \lambda_3 \mathcal{A}_3$$

and obtain the phase present at that minimum

4_ Target Space



CSL : $P_2^{(4)} = \frac{1}{3} P_1^{(4)} \Rightarrow A_2 = \frac{1}{3}$

$P_3^{(4)} = \frac{1}{3} P_1^{(4)} \Rightarrow A_3 = \frac{1}{3}$

Polar : $P_2^{(4)} = P_1^{(4)} \Rightarrow A_2 = 1$

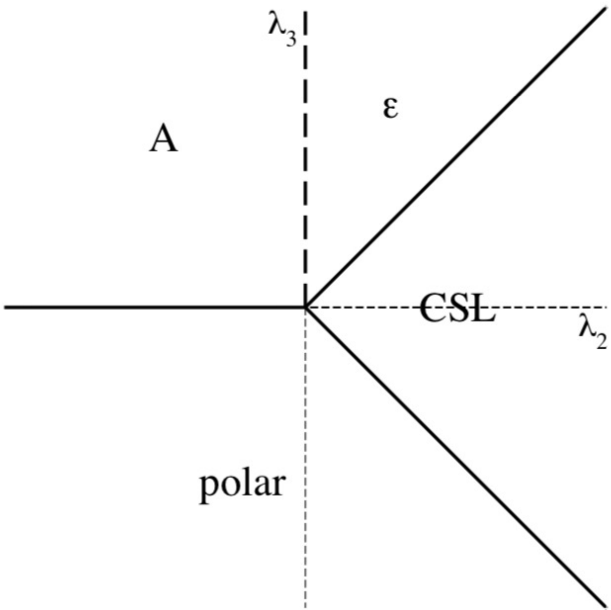
$P_3^{(4)} = P_1^{(4)} \Rightarrow A_3 = 1$

A : $P_3^{(4)} = 0 \Rightarrow A_3 = 0$

$P_2^{(4)} = P_1^{(4)} \Rightarrow A_2 = 1$

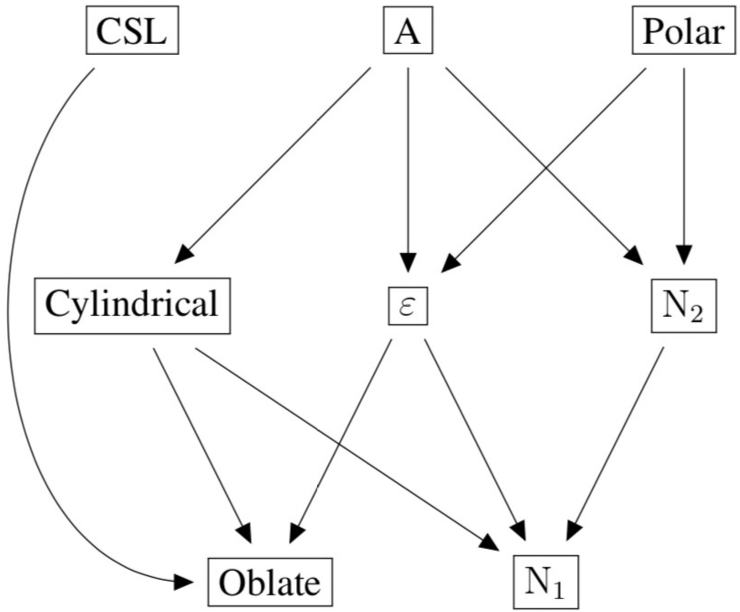
- There are 3 inert states that occupy the vertices and 1 non-inert state.
- AX line is concave, other lines are straight $\Rightarrow$ inequalities

5 _ Phase Diagram



- CSL : $\lambda_2 > \lambda_3 \quad \wedge \quad \lambda_2 + \lambda_3 > 0$
- Polar : $\lambda_3 < 0 \quad \wedge \quad \lambda_2 + \lambda_3 < 0$
- A : $\lambda_2 < 0 \quad \wedge \quad \lambda_3 > 0$
- ε : $\lambda_3 > \lambda_2 > 0$

6 _ Final Considerations



- A, CSL and Polar have maximal isotropy subgroups

- $\mathcal{E}$ present in the phase diagram and does not have maximal isotropy subgroup

$\Downarrow$

Contradiction of Michel's conjecture

- Isotropy subgroup of $\mathcal{E}$ is a subgroup of the isotropy subgroup of A $\Rightarrow$ second order phase transition

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Annex:

① In the last two references more details can be found about G-invariant functions and G-invariant polynomials here I will just make a small summary.

I start by presenting the reason of considering an Irreducible representation of the symmetry group G and then I will explain a little bit more the G-invariant polynomials.

• One is interested in considering the irreducible representation of G Since Michel's conjecture is relative to this case and this example was somehow a counterexample to this conjecture so one had to respect the conditions of the conjecture. Also considering an irreducible representation of G assumes, as Michel's explains, that the polynomials don't have an extremum in the stratum of the Δ that has the smallest conjugation class of the isotropy subgroup excluding a "maximal symmetry breaking" for the representation.
More details about Michel's conjecture and also the effects on the G-invariant polynomials of considering this type of representation can be found in Michel's paper which is the second to last in the references.

Now I will explain a little bit more about G-invariant functions like the potential of the system and the G-invariant polynomials more details about this can be seen in the last two references.

- As it was explained before orbit spaces are useful to study functions that are invariant under the transformation of a compact symmetry group G, these invariant functions are constant on the orbits and so may be studied using functions defined on the orbit space of the group action

In the specific example that is being considered: the spin one color superconductor $G = U(3)_L \times SO(3)_R$ is the compact symmetry group which acts linearly on a manifold $M \cong \mathbb{R}^{18}$

The polynomial functions defined in $\mathbb{R}^{18}$ form an algebra indicated with $\mathbb{R}[\mathbb{R}^{18}]$. G acts naturally on $\mathbb{R}[\mathbb{R}^{18}]$:

$$(gp)(x) = p(g^{-1}x) \qquad g \in G, \ \forall p \in \mathbb{R}[\mathbb{R}^{18}], \ x \in \mathbb{R}^{18}$$

$p \in \mathbb{R}[\mathbb{R}^{18}]$ is G-invariant if $gp = p \ \forall g \in G$. The G-invariant real polynomial functions on $\mathbb{R}^{18}$ form an algebra indicated by $\mathbb{R}[\mathbb{R}^{18}]^G$

By Hilbert's theorem on invariants, $\mathbb{R}[\mathbb{R}^{18}]^G$ is finitely generated, that is there exists $p_1, \dots, p_q \in \mathbb{R}[\mathbb{R}^{18}]^G$ such that $\forall\, p \in \mathbb{R}[\mathbb{R}^{18}]^G$ there is a unique polynomial $\hat{p}$ in q indeterminates such that:

$$p(x) = \hat{p}\,(p_1(x), \dots, p_q(x)) \qquad \forall\, x \in \mathbb{R}^{18}$$

The invariant polynomials $p_1, \dots, p_q$ are called basic invariant polynomials (these are the P_α considered during the presentation), they can be choosen to be real and homogeneous.

These basic polynomials separate the orbits in the sense that given two different orbits at least one of the q basic polynomials takes a different value on the two orbits. So this means that the orbit map:

$$\begin{aligned} \bar{p} :\ & \mathbb{R}^{18} \to \mathbb{R}^{q} \\ & x \to \bar{p}(x) = (p_1(x), \dots, p_q(x)) \end{aligned}$$

maps $\mathbb{R}^{18}$ into a subset $S \subset \mathbb{R}^q$ in a way that there is a one to one correspondence between orbits in $\mathbb{R}^{18}$ and points in S, there is a diffeomorphism between the orbit space $\mathbb{R}^{18}/G$ and the set S and for this reason S can be identified with the space of orbits.

Every G-invariant function such as the potential of the physical system may be expressed, then, in a unique way as a polynomial function of the G-invariant polynomials : $p_1, \dots, p_q$, hence the form of the potential presented during the presentation.

The degree of these polynomials is determined by the group G, also the form they have depends on the representation of the group G.

The number of polynomials considering the diffeomorphism between the space of orbits and S, then the number of the invariant polynomials is just given by the dimension of the space of orbits In order to compute the dimension of the space of orbits I will use the following geometry theorem:

Thm: Let G be a compact Lie group acting smoothly on the manifold M such that M/G is connected. If P is a principal orbit (an orbit of maximum dimension) then

$$\dim(M/G) = \dim(M) - \dim(P)$$

Now in this case since $M \cong \mathbb{R}^{18}$ then $\dim(M) = 18$, also in order to find the dimension of the orbit of maximum dimension one just needs to subtract to the dimension of the group the dimension of the minimum isotropy subgroup. Now the minimum isotropy subgroup is the trivial subgroup $\{1\}$ which corresponds to Δ having all entries different from zero, it doesn't appear on the table since there aren't any ground states with this isotropy subgroup. However considering this minimum isotropy subgroup its dimension is zero and so the dimension of the orbit of maximum dimension will just be the dimension of the group G:

$$G = U(3)_L \times SO(3)_R$$

$$\dim(G) = \dim(U(3)) + \dim(SO(3))$$

$$= (3)^2 + \frac{3(3-1)}{2} = 12$$

This then means that $\dim(P) = \dim(G) = 12$

By the theorem I presented above then I have that:

$$\dim(\mathbb{R}^{18}/G) = \dim(\mathbb{R}^{18}) - \dim(P)$$
$$= 18 - 12 = 6$$

Therefore the dimension of the space of orbits is 6.

In the potential that was presented there were only 3 invariant polynomials these will therefore not be enough to parameterized the space of orbits. These invariant polynomials will however define a non-injective map from the space of orbits to $\mathbb{R}^3$, after introducing the Angles A_2, A_3 it reduces to $\mathbb{R}^2$.

The article that I mainly followed for the whole presentation, which is the second in the references, states without demonstrating that in the spin one color superconductor there are only 3 invariant polynomials independent of fourth-order. This does not mean that there are only 3 invariant polynomials but rather that there are only 3 independent of fourth-order. As Michel's explains in his article the invariant polynomials of fourth-order are (for most of the systems) the only important polynomials to explore symmetry breaking.

(2) $SU(3) \times U(1) \cong U(3)$

Let's take the following surjective homomorphism:

$$\psi : SU(3) \times U(1) \longrightarrow U(3)$$
$$(u, z) \longrightarrow u \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix}$$

By the isomorphism theorem one has that $(SU(3) \times U(1))/\ker\psi$ is isomorphic to $U(3)$ So one needs to compute the kernel of this homomorphism

$$\ker\psi = \left\{ (u, z) \in SU(3) \times U(1) : u \cdot \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} = I \right\}$$

$$U \cdot \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \iff \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} = U^{+}$$

$\Downarrow$

Applying the determinant to both sides of this equation.

$$z^3 = \det U^{+}$$

$$SU(3) = \{ U \in U(3) : \det U = 1 \} \qquad = 1$$

- Therefore z is the 3-th root of 1. This then means that

$$\boxed{\ker \psi \cong \mathbb{Z}_3}$$

Then by the isomorphism theorem

$$\tilde{\varphi} : \quad SU(3) \times U(1) / \mathbb{Z}_3 \longrightarrow U(3)$$

Where $\tilde{\varphi}$ is an isomorphism. The following diagram is commutative :

$$\begin{tikzcd}
SU(3) \times U(1) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U(3) \\
SU(3) \times U(1) / \mathbb{Z}_3 \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

$\blacksquare$

3 $U(3)_L$ has nontrivial isotropy subgroups if and only if the matrix Δ has zero modes.

Specifically the isotropy subgroup will be $U(n)_L$ where n is the number of zero modes of Δ.

(4) SO(2) is isomorphic to a subgroup of SO(3) so one can say that SO(2) is a "subgroup" of SO(3).
Let's consider the following:

$$\Psi : SO(2) \longrightarrow SO(3)$$

$$R \longrightarrow \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

This is clearly an homomorphism (with matrix multiplication) since $\Psi(R_1 R_2) = \Psi(R_1) \cdot \Psi(R_2)$. It is bijective so it is an isomorphism.

Let's take $R \in SO(2)$ $\quad R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then:

$$\begin{cases} R R^T = I \\ \det R = 1 \end{cases}$$

Let's take one element of SO(2) :

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Now let's see what is the form that the order parameter Δ takes so that it leaves SO(2) unbroken (I will do reverse engineering to check the form of Δ):

$$\psi(R^T)\, \Delta\, \psi(R) = \Delta$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta_1 & ia_3 & -ia_2 \\ -ia_3 & \Delta_2 & ia_1 \\ ia_2 & -ia_1 & \Delta_3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \Delta_1 & ia_3 & -ia_2 \\ -ia_3 & \Delta_2 & ia_1 \\ ia_2 & -ia_1 & \Delta_3 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \Delta_2 & ia_3 & ia_1 \\ -ia_3 & \Delta_1 & ia_2 \\ -ia_1 & -ia_2 & \Delta_3 \end{bmatrix} = \begin{bmatrix} \Delta_1 & ia_3 & -ia_2 \\ -ia_3 & \Delta_2 & ia_1 \\ ia_2 & -ia_1 & \Delta_3 \end{bmatrix}$$

From this last equation one has that:

$$\begin{cases} \Delta_2 = \Delta_1 \\ a_3 = a_3 \\ a_1 = -a_2 \\ a_1 = a_2 \\ \Delta_3 = \Delta_3 \end{cases} \Rightarrow \quad 2a_2 = 0 \Rightarrow a_2 = 0 \Rightarrow a_1 = 0$$

Since $a_2 = a_1 = 0$ and $\Delta_2 = \Delta_1$ then I can rename the variables:

$$\begin{aligned} \Delta_1 = \Delta_2 &\longrightarrow \Delta_1 \\ \Delta_3 &\longrightarrow \Delta_2 \\ a_3 &\longrightarrow a \end{aligned}$$

Then the form of the order parameter is:

$$\Delta = \begin{bmatrix} \Delta_1 & ia & 0 \\ -ia & \Delta_1 & 0 \\ 0 & 0 & \Delta_2 \end{bmatrix}$$

Which is the one on the table

5. In order to explain what happens in the first line of the table above let's make an analogy:

- Consider the action of S^1 on the sphere that just rotates the sphere. It is pretty clear that the isotropy subgroup of all the points except the poles is just the trivial subgroup whereas the isotropy subgroup of the north and south poles is S^1.

Now consider that besides S^1 there is also another group G acting on the sphere such that it has different isotropy subgroups for the north and south pole. Then it is necessary to study the poles individually. What is happening in the first line is precisely this, despite having as an isotropy group $SO(2)_V$ the different Δ presented have different nontrivial isotropy subgroups of $U(3)_L$.

$\Rightarrow$ In the second line they all have $\vec{a} = \vec{0}$ so $\Delta = S$ in terms of the isotropy subgroups of $SO(3)_V$. The last two in this line N_1 and N_2 completely break the $SO(3)_V$ symmetry.

⇒ Finally looking at the isotropy subgroup of the cylindrical and $\mathcal{E}$ phase (or the phase A and polar) one may ask why they are not the same phase since their isotropy subgroup is the "same".

The answer here is actually precisely in the meaning of same, indeed in both cases the isotropy subgroups are isomorphic, however as it was explained during the presentation the table shows the different stratum and in order for two different Δ to be in the same stratum their isotropy subgroup have to be not only isomorphic but conjugate as well

There are three options.

- The two Δ can be in the same orbit and therefore they have conjugate isotropy subgroups and the conjugacy defines an isomorphism between the subgroups (natural isomorphism)

- The two Δ can be in different orbits however their isotropy subgroups are conjugate of one another. These Δ and the ones from the point above are all in the same stratum

- The two Δ can have isomorphic isotropy subgroups however the two Δ are not in the same orbit and their isotropy subgroups are not conjugate of one another (accidental isomorphism).

$$\Downarrow$$

If the Δs are being grouped depending on wether they are on the same stratum then in this case they are considered two different Δs

$\Rightarrow$ This last case is precisely what happens with the cylindrical and $\mathcal{E}$ but also with A and polar.

⑥ None of the phases A, polar, CSL provide a minimum of the potential for $\lambda_3 > \lambda_2 > 0$

These would be the candidates according to Michel's theorem, since these are the inert states.

Let's then assume:

$$P_3^{(4)} \leq \frac{1}{9} P_1^{(4)}$$

Using theorem 6 and the Cauchy inequality with

$$u_1 = \sqrt{\lambda_2 \left(P_2^{(4)} - P_3^{(4)}\right)} \qquad u_2 = \sqrt{(\lambda_2 + \lambda_3) P_3^{(4)}}$$

$$v_1 = \frac{1}{\sqrt{\lambda_2}} \qquad v_2 = \frac{1}{\sqrt{\lambda_2 + \lambda_3}}$$

$$\lambda_2 P_2^{(4)} + \lambda_3 P_3^{(4)} = \lambda_2 \left(P_2^{(4)} - P_3^{(4)}\right) + (\lambda_2 + \lambda_3) P_3^{(4)}$$

$$= u_1^2 + u_2^2$$

$$\geq \frac{(u_1 v_1 + u_2 v_2)^2}{v_1^2 + v_2^2}$$

$$= \frac{\left(\sqrt{P_2^{(4)} - P_3^{(4)}} + \sqrt{P_3^{(4)}}\right)^2}{\frac{1}{\lambda_2} + \frac{1}{\lambda_2 + \lambda_3}} \geq \frac{\lambda_2(\lambda_2 + \lambda_3)}{2\lambda_2 + \lambda_3} P_1^{(4)}$$

- In order to have equality : $P_1^{(4)} = \left(\sqrt{P_2^{(4)} - P_3^{(4)}} + \sqrt{P_3^{(4)}}\right)^2$

$\Rightarrow$ By theorem 6 phase ε.